

Multi-indexed Extensions of Soliton Potential and Extended Integer Solitons of KdV Equation

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Abstract

We calculate infinite set of initial profiles of higher integer KdV solitons, which are both exactly solvable for the Schrodinger equation and for the Gel'fand-Levitan-Marchenko equation in the inverse scattering transform method of KdV equation. The calculation of these higher integer soliton solutions is based on the recently developed multi-indexed extensions of the reflectionless soliton potential.

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I. INTRODUCTION

One well-known class of soliton of interest in both physics and mathematics is the non-topological soliton described by the Korteweg-de Vries (KdV) equation, i.e.,

$$u_t - 6uu_x + u_{xxx} = 0 \quad (1.1)$$

in one space $x \in (-\infty, \infty)$ and one time $t > 0$ dimension. There are many physical systems which are weakly dispersive and weakly nonlinear that can be well described by KdV equation. The phenomena of blood pressure waves [1], the internal solitary waves in oceanography [2] observed in the Andaman sea and the nonlinear electrical chains etc. are among some of them. Thus it is of interest to solve and better understand KdV equation from different angle and in different context. Of various approaches to solving nonlinear partial differential equation such as the KdV equation, the method of the inverse scattering transform (IST) [3, 4] invented in 1960's is one of the most important development on this subject. According to the method of IST, the solution of the KdV equation is converted to the solution of two simpler linear equations, namely, the quantum mechanical Schrödinger equation and the Gel'fand-Levitan-Marchenko (GLM) equation [5, 6].

For soliton solutions the related Schrödinger equation is connected with reflectionless potentials [7]. For such reflectionless potentials, the reflection amplitudes of the scattering

states vanish, and the corresponding GLM equation is easy to solve. One gets $2N$ continuous parameters, N norming constants $c_n(0)$ and N energy parameters κ_n , for the general N -soliton solution. Furthermore, as it turns out, only the κ_n parameters survive asymptotically as $t \rightarrow \pm\infty$. These N parameters fix the amplitudes, speeds and the relative phases of the bumps of the solitons.

In this work we would like to point out a denumerably infinite set of higher integer soliton solutions of the KdV equations. The initial profiles of these solutions are related to the recently discovered exactly solvable quantum mechanical systems [8–12], based on multiple Darboux-Crum transformations [13–15]. Such transformations can generate new solvable quantum systems from the previous known ones using certain polynomial type seed solutions. These seed functions are called the virtual and pseudo virtual state wavefunctions [8, 16, 17]. They were obtained from the eigenfunctions by discrete symmetry operations or by using the same functional forms of the eigenfunctions with their degrees higher than the highest eigenlevel (these are called the over-shooting states). The one-indexed [18] and more complete multi-indexed extensions [19] of the known quantum scattering problems [20] were recently calculated along this line of thoughts.

The Darboux-Crum transformation in terms of the pseudo virtual state will generate a new bound state below the original ground-state. Therefore it generates a non-isospectral deformation. In this paper we will use pseudo virtual state wavefunction to deform soliton potential with positive integer parameter h . We will obtain an infinite number of reflectionless potentials, which can be served as the initial profiles of integer KdV solitons in the inverse scattering method mentioned above. Although the profiles we obtained are not new soliton solutions, the method we adopted based on recently developed multi-indexed extensions of the reflectionless soliton potential to systematically generate higher integer KdV solitons is interesting and, most importantly, mathematically simpler and more effective.

II. SOLVABLE HIGHER INTEGER 2-SOLITON $(\kappa_0, \kappa_1) = (1, 4)$

We begin with a specific example of a solvable 2-soliton potential, namely, the simplest 1-step deformed soliton potential under Darboux-Crum transformation. The scattering data, or the bound state problem and the scattering problem, of this potential was recently calculated in [19]. The bound state problem and the scattering problem of the original soliton

potential

$$U(x) = -\frac{h(h+1)}{\cosh^2 x} = -h(h+1) \operatorname{sech}^2 x, \quad h > 0, \quad -\infty < x < \infty \quad (2.2)$$

can be found in [7, 20]. This potential contains finitely many bound states

$$\begin{aligned} \phi_n(x) &= \frac{1}{(\cosh x)^{h-n}} P_n^{(h-n, h-n)}(\tanh x) \\ &\sim \frac{1}{(\cosh x)^{h-n}} {}_2F_1 \left(-n, 2h-n+1, h-n+1, \frac{1-\tanh x}{2} \right), \\ E_n &= -\kappa_n^2 = -(h-n)^2; \quad n = 0, 1, 2, \dots, [h]', \end{aligned} \quad (2.3)$$

where $P_n^{(h-n, h-n)}(x)$ is the Jacobi polynomial and ${}_2F_1(x)$ is the hypergeometric function ($[h]'$ denotes the greatest integer not exceeding and not equal to h).

The soliton potential contains a discrete symmetry

$$h \rightarrow -(h+1), \quad (2.4)$$

which can be used to construct the seed function

$$\varphi_v(x) = (\cosh x)^{h+1+v} P_\nu^{(-h-1-v, -h-1-v)}(\tanh x), \quad v = 0, 1, 2, 3, 4, \dots \quad (2.5)$$

with energy

$$E_v = -(h+1+v)^2. \quad (2.6)$$

It turns out that for $v = 1, 3, 5, \dots$, the deformed potential contains pole at $x = 0$. For example, for $v = 1$,

$$U_1(x) = U(x) - 2 \frac{d^2}{dx^2} \log \varphi_1(x) = U(x) - \frac{2(h+1)}{\cosh^2 x} + \frac{2}{\sinh^2 x} \quad (2.7)$$

which contains pole at $x = 0$. We note that although one can define the asymptotic forms of the scattering state for this potential, the corresponding bound state wavefunctions contain singularities. So for our purpose here, only $v = 2, 4, 6, \dots$ can be used to deform the soliton potential.

For simplicity and clarity of presentation, we will first use the seed function for $h = 1$ ($n = 0$) and $v = 2$

$$\varphi_2(x)_{h=1} = \frac{1}{2} \cosh^4 x (1 + 5 \tanh^2 x) \quad (2.8)$$

to illustrate the calculation. The deformed potential is easily calculated to be

$$U_2(x)_{h=1} = U(x) - 2 \frac{d^2}{dx^2} \log \varphi_2(x)_{h=1} = - \frac{30(4 \cosh^4 x - 8 \cosh^2 x + 5)}{\cosh^2 x (36 \cosh^4 x - 60 \cosh^2 x + 25)} \quad (2.9)$$

which has no pole and no zero for the whole regime of x and approaches 0 asymptotically for $x \rightarrow \pm\infty$ as $U(x)_{h=1}$ does. Note that $U_2(x=0)_{h=1} - U(x=0)_{h=1} = -28 < 0$, which suggests the existence of a lowest new bound state for the deformed potential $U_2(x)_{h=1}$. The bound state wavefunctions of the deformed potential Eq.(2.9) can be calculated through the Darboux-Crum transformation to be

$$\psi_0(x) = \phi'_0 - \frac{\varphi'_2}{\varphi_2} \phi_0 = -5 \operatorname{sech} x \tanh x \left(1 + \frac{2 \operatorname{sech}^2 x}{(1 + 5 \tanh^2 x)} \right) \quad (2.10)$$

with energy

$$E_0 = -\kappa_0^2 = -(h-n)^2 = -1. \quad (2.11)$$

It can be easily shown that there is another bound state of the deformed potential

$$\psi_1(x) \sim \frac{1}{\varphi_2} = \frac{2}{\cosh^4 x (1 + 5 \tanh^2 x)} \quad (2.12)$$

with a lower energy

$$E_1 = -\kappa_1^2 = -(h+1+v)^2 = -4^2 \quad (2.13)$$

as was expected previously. The normalized wavefunctions and their asymptotic forms can be calculated to be

$$\psi_0(x) = \sqrt{\frac{15}{2}} \operatorname{sech} x \tanh x \left(1 + \frac{2 \operatorname{sech}^2 x}{(1 + 5 \tanh^2 x)} \right) \rightarrow \sqrt{\frac{10}{3}} e^{-x} \text{ as } x \rightarrow \infty, \quad (2.14)$$

$$\psi_1(x) = \sqrt{\frac{15}{8}} \frac{2}{\cosh^4 x (1 + 5 \tanh^2 x)} \rightarrow \sqrt{\frac{40}{3}} e^{-x} \text{ as } x \rightarrow \infty. \quad (2.15)$$

The constants

$$c_0(0) = \sqrt{\frac{10}{3}}, \quad c_1(0) = \sqrt{\frac{40}{3}} \quad (2.16)$$

in equations Eq.(2.14) and Eq.(2.15) are called norming constants. The reflection amplitude of the scattering of the M -step ($M = 1$ for the present case) deformed soliton potential Eq.(2.9) was calculated to be [19]

$$r_D(k) = r(k) \cdot \prod_{j=1}^M (-)^j \frac{k + i(h + v_j + 1)}{k - i(h + v_j + 1)}, \quad (2.17)$$

where

$$r(k) = \frac{\Gamma(1+h-ik)\Gamma(-h-ik)\Gamma(ik)}{\Gamma(-h)\Gamma(1+h)\Gamma(-ik)} \quad (2.18)$$

is the reflection amplitude for the undeformed potential in Eq.(2.2). In view of the multiplicative form of $r_D(k)$, it is important to note that, for integer $h = 1, 2, 3, \dots$, the scattering of the deformed potential remains reflectionless as the undeformed potential due to the factor $\Gamma(-h)$ in the denominator of $r(k)$.

We are now ready to use the scattering data $\{\kappa_n, c_n, r_D(k)\}$ to solve the KdV equation. For the reflectionless potential, $r_D(k) = 0$, the GLM equation is easy to solve, and the solution $u(x, t)$ is given by [4]

$$u(x, t) = -2 \frac{d^2}{dx^2} \log(\det A), \quad (2.19)$$

where A is a $N \times N$ matrix ($N \equiv h + 1$) with elements A_{mn} given by

$$A_{mn} = \delta_{mn} + c_n^2(t) \frac{\exp -(\kappa_m + \kappa_n)x}{\kappa_m + \kappa_n}; \quad m, n = 0, 1, 2, \dots, N-1. \quad (2.20)$$

In Eq.(2.20) $c_n(t) = c_n(0) \exp(4\kappa_n^3 t)$ and is one of the Gardner-Greene-Kruskal-Miura (GGKM) equations [3].

For the present case, $N = h + 1 = 2$. The integer 2-soliton solution corresponding to $(\kappa_0, \kappa_1) = (1, 4)$ can be calculated to be

$$u(x, t)_{(1,4)} = -\frac{120e^{8t+2x}(e^{1024t} + e^{16x} + 16e^{520t+6x} + 30e^{512t+8x} + 16e^{504t+10x})}{(3e^{520t} + 3e^{10x} + 5e^{512t+2x} + 5e^{8t+8x})^2}. \quad (2.21)$$

By taking $t = 0$ in Eq.(2.21), one reproduces the initial profile $u(x, 0) = U_2(x)_{h=1}$ calculated in Eq.(2.9). The asymptotic form of the $(\kappa_0, \kappa_1) = (1, 4)$ solution is

$$u(x, t)_{(1,4)} \sim -2 \sum_{n=0}^{N-1} \kappa_n^2 \sec h^2 \{ \kappa_n(x - 4\kappa_n^2 t) \pm \chi_n \}, t \rightarrow \pm\infty, \quad (2.22)$$

where

$$\exp(2\chi_n) = \prod_{\substack{m=0 \\ m \neq n}}^{N-1} \left| \frac{\kappa_n - \kappa_m}{\kappa_n + \kappa_m} \right|^{sgn(\kappa_n - \kappa_m)}. \quad (2.23)$$

Interestingly, it is seen that the asymptotic form of the solitary wave is independent of $c_n(0)$ and is determined solely by the eigenvalues κ_n 's. Note also that the previous integer 2-soliton solution corresponds to $(\kappa_0, \kappa_1) = (1, 2)$. We stress that the general 2-soliton solution contains four continuous parameters $\kappa_0, \kappa_1, c_0(0)$ and $c_1(0)$, and is given by Eq.(2.19) with

$$\det A = \left\{ 1 + \frac{c_0(t)^2}{2\kappa_0} e^{-2\kappa_0 x} \right\} \left\{ 1 + \frac{c_1(t)^2}{2\kappa_1} e^{-2\kappa_1 x} \right\} - \frac{c_0(t)^2 c_1(t)^2}{(\kappa_0 + \kappa_1)^2} e^{-2(\kappa_0 + \kappa_1)x}. \quad (2.24)$$

The $(\kappa_0, \kappa_1) = (1, 4)$ solution we obtained corresponds to discrete parameters with values given in Eq.(2.11), Eq.(2.13) and Eq.(2.16). The $(1, 4)$ integer soliton solution, similar to the previous $(1, 2)$ solution, is exactly solvable quantum mechanically. On the other hand, the scattering data obtained from, for example, $h = \frac{1}{2}$ ($n = 0$) and $v = 2$ is exactly solvable quantum mechanically, but the corresponding GLM equation is not solvable since the reflection amplitude is not zero. It is interesting to see that the calculation of these higher integer soliton solutions such as the $(1, 4)$ integer soliton is based on the recently developed multi-indexed extensions of the reflectionless soliton potential.

III. SOLVABLE HIGHER INTEGER N-SOLITONS

The result of section II can be generalized to higher solvable N -soliton cases (solvable in the sense of inverse scattering method). Here we present the result for 1-step deformation and take $v = 2$, $h = 1, 2, 3, 4, \dots$. The normalized bound state wavefunctions of the deformed potential

$$U_2(x)_h = U(x) - 2 \frac{d^2}{dx^2} \log \varphi_2(x) \quad (3.25)$$

can be calculated through the Darboux-Crum transformation to be

$$\begin{aligned} \psi_n(x) &= \frac{1}{\sqrt{B_n(E_n - E_h)}} \left(\phi'_n - \frac{\varphi'_2}{\varphi_2} \phi_n \right) \\ &= \frac{1}{\sqrt{B_n(E_n - E_h)}} \left\{ \frac{2h - n + 1}{2(\cosh x)^{h-n+2}} P_{n-1}^{(h-n+1, h-n+1)}(\tanh x) \right. \\ &\quad \left. - \left(\frac{(2h - n + 3) \tanh x}{(\cosh x)^{h-n}} + \frac{2(2h + 3) \tanh x}{[1 + (2h + 3) \tanh^2 x](\cosh x)^{h-n+2}} \right) \right. \\ &\quad \left. \times P_n^{(h-n, h-n)}(\tanh x) \right\}, \end{aligned} \quad (3.26)$$

$$B_n = \frac{2^{2(h-n)} \Gamma(h+1)^2}{n!(h-n)\Gamma(2h-n+1)} \quad (3.27)$$

with energy

$$E_n = -\kappa_n^2 = -(h-n)^2; n = 0, 1, 2, \dots, h-1. \quad (3.28)$$

In addition, there is a newly added bound state, given by $1/\varphi_2$. The normalized form of this state is

$$\psi_h(x) = \sqrt{\frac{2\Gamma(h+\frac{5}{2})}{\pi^{1/2}\Gamma(h+2)}} \frac{1}{(\cosh x)^{h+3}[1 + (2h+3)\tanh^2 x]} \quad (3.29)$$

with lowest energy

$$E_h = -\kappa_h^2 = -(h+1+v)^2 = -(h+3)^2. \quad (3.30)$$

By Eq.(2.17) the scattering of the deformed potential is reflectionless. The scattering data needed are

$$c_n(0) = \frac{1}{(h-n)!} \sqrt{\frac{(h-n)(2h-n+3)(2h-n)!}{(n+3)n!}}, n = 0, 1, 2, \dots, h-1, \quad (3.31)$$

$$c_h(0) = \frac{2^{h+2}}{h+2} \sqrt{\frac{2\Gamma(h+\frac{5}{2})}{\pi^{1/2}\Gamma(h+2)}}; \quad (3.32)$$

$$\kappa_n = (h-n), n = 0, 1, 2, \dots, h-1, \quad (3.33)$$

$$\kappa_h = h+3. \quad (3.34)$$

The general formula for the extended soliton solutions $u(x, t)$ is then obtained by Eq.(2.19) and Eq.(2.20) with $N = h+1$. By taking $t = 0$ in Eq.(2.19), one reproduces the initial profile $U_2(x)_h$ calculated in Eq.(3.25)

$$U_2(x)_h = -2 \frac{d^2}{dx^2} \log(\det A)_{t=0}. \quad (3.35)$$

The profile for $h = 2$, for example, is the extended solvable 3-soliton (1, 2, 5)

$$U_2(x)_{h=2} = u(x, 0)_{(1,2,5)} = -\frac{4(144 \cosh^4 x - 280 \cosh^2 x + 147)}{\cosh^2 x (64 \cosh^4 x - 112 \cosh^2 x + 49)}, \quad (3.36)$$

and

$$\begin{aligned} u(x, t)_{(1,2,5)} = & -(16e^{8t+2x}(9e^{2128t} + 9e^{28x} + 1575e^{16(63t+x)} + 882e^{16(66t+x)} + 3252e^{14(76t+x)} \\ & + 175e^{8(142t+x)} + 49e^{8(250t+x)} + 126e^{4(516t+x)} + 56e^{2072t+2x} + 126e^{2056t+6x} \\ & + 1008e^{1128t+10x} + 882e^{1072t+12x} + 1575e^{1120t+12x} + 1008e^{1000t+18x} + 49e^{128t+20x} \\ & + 175e^{992t+20x} + 126e^{72t+22x} + 126e^{64t+24x} + 56e^{56t+26x}) \\ & / (2e^{1072t} + 2e^{16x} + 14e^{4(252t+x)} + 9e^{2(532t+x)} + 7e^{1000t+6x} + 7e^{72t+10x} \\ & + 14e^{64t+12x} + 9e^{8t+14x})^2. \end{aligned} \quad (3.37)$$

IV. DISCUSSION

In this paper we have pointed out an infinite set of higher integer initial profiles of the KdV solitons, which are both exactly solvable for the Schrodinger equation and for the

Gel'fand-Levitan-Marchenko equation in the inverse scattering transform method of KdV equation. The calculation of these solutions are based on the multi-indexed extensions of the reflectionless soliton potential based on the Darboux-Crum transformation.

For simplicity and clarity of presentation, we have discussed only the case of 1-step extension using the pseudo-virtual states obtained by discrete symmetry with integral index $v = 2$. Our discussion can be straightforwardly extended to general values of even v , to the general M -step deformations with $M = N - h$, and to the cases using over-shooting pseudo-virtual states [16, 17, 19]. Eq.(2.17) ensures that the deformed potentials remain reflectionless. For these cases, one needs to take care of the singularity problem and avoid the singularities in the soliton profiles [15, 16, 19]. Thus for extended 3-solitons, for example, one could have two classes of solvable solitons. The first class is

$$\begin{aligned} \text{Class } I : N = 3, h = 2, M = 1 \\ (\kappa_0 = 1, \kappa_1 = 2, \kappa_2 = v_1 + 3) \\ v_1 = 2, 4, 6, 8, \dots \end{aligned} \quad (4.38)$$

and the second class is

$$\begin{aligned} \text{Class } II : N = 3, h = 1, M = 2 \\ (\kappa_0 = 1, \kappa_1 = v_1 + 2, \kappa_2 = v_2 + 2) \\ v_1 = 2, 4, 6, 8, \dots, v_2 - v_1 = 3, 5, 7, 9, \dots \end{aligned} \quad (4.39)$$

In general the initial profiles of the solvable $N = h + M$ solitons contain integer parameters $\{h, v_1, v_2, \dots, v_M\}$ and can be calculated as following. The undeformed soliton potential can be written as

$$-h(h+1) \operatorname{sech}^2 x = -2 \frac{d^2}{dx^2} \log(\det A)_{t=0} \quad (4.40)$$

where the functional form of $A_{t=0}$ is given by Eq.(2.20) with $N = h$, and

$$\kappa_n = h - n, c_n(0) = \frac{1}{(h-n)!} \sqrt{\frac{(h-n)(2h-n)!}{n!}}, n = 0, 1, 2, \dots, h-1. \quad (4.41)$$

The M -step deformed potential can be written as [17]

$$U(x)_{deformed} = U(x)_{undeformed} - 2 \frac{d^2}{dx^2} \log |W[\varphi_{v_1}, \varphi_{v_2}, \dots, \varphi_{v_M}]|.$$

where $W[\varphi_{v_1}, \varphi_{v_2}, \dots, \varphi_{v_M}]$ is the Wronskian of the seed functions $\{\varphi_{v_1}, \varphi_{v_2}, \dots, \varphi_{v_M}\}$. So the solvable deformed potentials or the initial profiles of the solitons discussed in this paper can be written as

$$\begin{aligned}
U(x)_{deformed} &= u(x, 0)_{\{h, v_1, v_2, \dots, v_M\}} = -2 \frac{d^2}{dx^2} \log(\det A)_{t=0} - 2 \frac{d^2}{dx^2} \log |W[\varphi_{v_1}, \varphi_{v_2}, \dots, \varphi_{v_M}]| \\
&= -2 \frac{d^2}{dx^2} \log\{(\det A)_{t=0} |W|\} = -2 \frac{d^2}{dx^2} \log\{(\det \hat{A})_{t=0} |\hat{W}|\} \\
&= -2 \frac{d^2}{dx^2} \log\{\det(\hat{A}_{t=0} \cdot \pm \hat{W})\} = -2 \frac{d^2}{dx^2} \log(\det \hat{A}'_{t=0}), \tag{4.42}
\end{aligned}$$

which is the generalization of Eq.(3.35). In Eq.(4.42), \hat{A} and \hat{W} are $N \times N$ matrices extended from lower $h \times h$ and $M \times M$ matrices without changing the values of \det . To obtain \hat{A} from A , for example, one adds M unit row (and column) vectors to A matrix to get a $N \times N$ matrix \hat{A} . For the first row (column), one adds $(1, 0, 0, \dots, 0)$; the second row (column), one adds $(0, 1, 0, 0, \dots, 0)$ etc. Similar adding can be done for the matrix W to get a $N \times N$ matrix \hat{W} . Finally, the resulting $N \times N$ matrix $\hat{A}'_{t=0}$ calculated in Eq.(4.42) can be reduced without changing the value of \det to the form of Eq.(2.20) with some values of integer κ'_n and real $c'_n(0)$ of the corresponding extended solvable N -soliton.

The existence of integer κ'_n and real $c'_n(0)$ in Eq.(4.42) are guaranteed since, in our approach, we know that $U(x)_{deformed}$ is a reflectionless solvable potential. So in this case Eq.(2.19) and Eq.(2.20) can be applied.

In the beginning of our calculation, we could have set $h = 0$ and did the say 2-step deformation. We then end up with, for $v_1 = 2$ for example, $(3, 6)$, $(3, 8)$, $(3, 10)$solitons.

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addressed the issue of reflectionless potentials from a different context.

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